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A Long-Time Tail for Random Walk in Random Scenery

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Consider a simple random walk on $\mathbb{Z}^d$ whose sites are colored black or white independently with probability q, resp. $1-q$. Walk and coloring are independent. Let n_k be the number of steps by the walk between its kth and $(k+1)$ th visits to a black site (i.e., the length of its kth white run), and let $\Delta_k = E(n_k) - q^{-1}$. Our main result is a proof that $(*)$ $\lim_{k\to\infty} k^{d/2}\Delta_k = (1-q)q^{d/2-2}(d/2\pi)^{d/2}$. Since it is known that $q^{-1}\Delta_k = E(n_1 n_{k+1} \mid B) - E(n_1 \mid B)\, E(n_{k+1} \mid B)$, with B the event that the origin is black, $(*)$ exhibits a long-time tail in the run length autocorrelation function. Numerical calculations of Δ_k $(1 \leqslant k \leqslant 100)$ in $d = 1, 2$, and 3 show that there is an oscillatory behavior of Δ_k for small k. This damps exponentially fast, following which the power law sets in fairly rapidly. We prove that if the coloring is not independent, but is convex in the sense of FKG, then the decay of Δ_k cannot be faster than $(*)$.

KEY WORDS: Random walk in random scenery; interarrival times; run length autocorrelation function; long-time tail; FKG inequality; local times.

1. INTRODUCTION AND STATEMENT OF RESULTS

This paper treats a model of a random walk on a lattice with a random black–white coloring assigned to its sites. Walk and coloring are independent. We are interested in the asymptotic behavior of the interarrival times between successive visits of the walk to the black sites, in particular, in the occurrence of a *long-time tail* in the averages and autocorrelations of the interarrival times.

Long-time tails have been found in many related models describing diffusion in disordered media, e.g., Lorentz models, trapping models, and hydrodynamic fluid models (see ref. 1 for a review; also see refs. 2). With

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few exceptions, the theoretical analysis of such models is hard. Not only are most of the arguments appearing in the literature heuristic, they often fail to give an explicit evaluation of the tail amplitude, especially in higher dimensions. Numerical simulations of long-time tails are generally difficult to perform accurately and often agree only qualitatively with the theoretical predictions. The model in the present paper is relatively simple because walk and coloring are assumed to be independent. This will allow us to prove a rigorous and explicit long-time tail result, and to develop an accurate numerical analysis as well.

Most of what we shall have to say will refer to a Bernoulli coloring and a simple random walk, but it will pay to start off in a more general setting. Throughout the paper we shall use the symbols P and E to denote probability and expectation with respect to either coloring or walk or both.

Consider the lattice $\mathbb{Z}^d$ and associate with it:

(i) A *random coloring* $(C(z))_{z \in \mathbb{Z}^d}$ with $C(z) \in \{B, W\}$ for each z.

(ii) A *random walk* $(X_n)_{n \geqslant 0}$ with $X_n \in \mathbb{Z}^d$ for each n and $X_0 = 0$.

Assume that the coloring is *stationary* and *ergodic* with

$$0 < q = P(C(0) = B) < 1$$

and that the increments $X_{n+1} - X_n$, $n \geqslant 0$, of the walk are *i.i.d.* and *aperiodic*. The sequence $(C(X_n))_{n \geqslant 0}$ of colors hit by the walk is stationary and ergodic,[3] and determines the sequence $(T_k)_{k \geqslant 1}$ of black hitting times,

$$\begin{aligned} C(X_n) &= B \qquad \text{for} \quad n = T_1, T_2, \ldots \\ C(X_n) &= W \qquad \text{otherwise} \end{aligned}$$

Our main object of study is the sequence $(n_k)_{k \geqslant 0}$ of interarrival times between black hits

$$\begin{aligned} n_0 &= T_1 \\ n_k &= T_{k+1} - T_k \qquad (k \geqslant 1) \end{aligned}$$

to which we shall refer as *run lengths*.

There is an anomalous effect (well known in renewal theory) which causes the n_k for different k to have different distributions. This is seen most easily by considering a Bernoulli coloring and a simple random walk on $\mathbb{Z}$. The white interval containing 0 is stochastically larger than the other white intervals. Because the walk may return to the origin many times, it tends to spend an anomalously long time in this interval, causing the n_k to be stochastically larger than expected. As $k \to \infty$, the walk diffuses away and the n_k decrease to a limiting distribution. The latter corresponds to the walk seeing a stationary color scenery at the beginning of its white runs. It is precisely this decay that we want to capture.

It has been shown[4] that if the color distribution is tail trivial (which includes, e.g., all extremal Gibbs states for a given interaction), then for each integer $m \geqslant 0$

$$\lim_{k \to \infty} P(n_k > m) = q^{-1} P(n_0 = m) \tag{1}$$

This expresses the limiting run length distribution in terms of the distribution of n_0, the length of the initial run to a black site. Although very little is known about n_0 in general,[3] we can at least infer from (1) that

$$\lim_{k \to \infty} E(n_k) = q^{-1} \tag{2}$$

provided we can come up with a suitable dominated convergence argument.[4] So it is natural to consider the quantities

$$\Delta_k = E(n_k) - q^{-1} \qquad (k \geqslant 1)$$

which are the average "excess" run lengths. The following identities[6] will be needed later on

$$q^{-1} = E(n_k \mid B) \qquad (k \geqslant 1) \tag{3}$$

$$q^{-1} E(n_k) = E(n_1 n_{k+1} \mid B) \qquad (k \geqslant 1) \tag{4}$$

Here B is a short-hand notation for the event $\{C(0) = B\}$. Combining (3) and (4), we have

$$\begin{aligned} \Delta_k &= E(n_k) - E(n_k \mid B) \\ &= q[E(n_1 n_{k+1} \mid B) - E(n_1 \mid B)\, E(n_{k+1} \mid B)] \qquad (k \geqslant 1) \end{aligned} \tag{5}$$

[3] For Bernoulli coloring there are several asymptotic results known for n_0, e.g., $P(n_0 = m)$ for large m and $E(n_0)$ for small q. See, e.g., refs. 5.

[4] By Cauchy–Schwarz,

$$\begin{aligned} q^{-1} P(n_k > m) &= E(n_1 1_{\{n_{k+1} > m\}} \mid B) \\ &\leqslant [E(n_1^2 \mid B)\, P(n_{k+1} > m \mid B)]^{1/2} \\ &= q^{-1}[(1 + 2E(n_0))\, P(n_0 = m)]^{1/2} \end{aligned}$$

for all $k \geqslant 1$ and $m \geqslant 0$. The two equalities follow from refs. 6 and 7. Hence (1) implies (2) when $\sum_{m \geqslant 0} [P(n_0 = m)]^{1/2} < \infty$, which can be shown to hold under a weak mixing condition for the color distribution. Then also $E(n_0) < \infty$ because $P(n_0 = m)$ is monotone in m.[6]

Thus we see that Δ_k also manifests itself as a *run length autocorrelation* function. It is known that the sequence $(n_k)_{k \geqslant 1}$ given B is stationary.[6,7]

In Sections 2 and 3 we prove the following three theorems. The first gives an inequality valid for all color distributions which are *convex in the sense of FKG.*[8] The second gives an identity valid for Bernoulli coloring. The third is our long-time tail result.

Theorem 1. Suppose that the color distribution is convex. Then for arbitrary random walk, any $0 < q < 1$, and any $0 \leqslant \eta < 1$,

$$E(n_0) + \sum_{k \geqslant 1} \eta^k \Delta_k \geqslant \frac{1-q}{q} G(1 - q + q\eta) \tag{6}$$

Here $G(z) = \sum_{n \geqslant 0} z^n P(X_n = 0)$ is the generating function for return to the origin by the random walk.

Theorem 2. Suppose that the color distribution is Bernoulli. Then for arbitrary random walk and any $0 < q < 1$,

$$E(n_0) + \lim_{\eta \uparrow 1} \sum_{k \geqslant 1} \eta^k \Delta_k = \frac{1-q}{q} G(1) \tag{7}$$

This is finite if and only if the random walk is transient.

Theorem 3. Suppose that the color distribution is Bernoulli. Then for simple random walk and any $0 < q < 1$,

$$\lim_{k \to \infty} k^{d/2} \Delta_k = (1-q) q^{d/2-2} (d/2\pi)^{d/2} \tag{8}$$

Theorem 1 can be interpreted as an inequality for the mean trapping time $\sum_{k \geqslant 0} \eta^k E(n_k)$ when the black sites act as traps with survival probability η. Its generality is interesting because most results in the trapping literature are restricted to Bernoulli distribution. Theorem 2 says that, for Bernoulli coloring, (6) reduces to an equality as $\eta \uparrow 1$. As to Theorem 3, for simple random walk $G(1 - q + q\eta) = \sum_{k \geqslant 0} \eta^k g_k$ with $k^{d/2} g_k \to q^{d/2-1} (d/2\pi)^{d/2}$ as $k \to \infty$ (see Section 3). So (8) says that, for Bernoulli coloring and simple random walk, both sides of (6) have the same asymptotic coefficients and hence the same asymptotic behavior as $\eta \uparrow 1$. From (8) it follows that the Δ_k are asymptotically positive, in which case the limit and the sum in (7) may be interchanged. Incidentally, note that (7) is a rather remarkable identity because there are no exact expressions known for any of the expectations appearing under the sum.[5]

[5] For simple random walk in $d = 1$ one can calculate $E(n_0)$ and the first few Δ_k by hand, but there is no exact expression known for general k even for this simple case. The first few Δ_k turn out to be monotone decreasing in k, suggesting that possibly Δ_k is monotone for all k. The simulations in Section 4 seem to support this. The Δ_k are not monotone in $d \geqslant 2$.

To explain heuristically where the right-hand side of (8) comes from, let us note that, by (3), $\Delta_k = (1-q)[E(n_k \mid W) - E(n_k \mid B)]$ with W the event $\{C(0) = W\}$. First, the idea is that the following should be true as $k \to \infty$:

$$E(n_k \mid W) - E(n_k \mid B) \sim q^{-1} E(|\{T_k < n \leqslant T_{k+1}: X_n = 0\}|) \tag{9}$$

Indeed, if the origin is white, then each time the walk visits the origin during its kth run the expected time to complete this run will be prolonged by an amount q^{-1} compared to when the origin is black. This is precisely what is expressed by (3), q^{-1} being equal to the expected time needed to either return to the origin or hit a black site outside the origin. Next, since by (2) the expected length of the kth run converges to q^{-1}, it should also be true that as $k \to \infty$,

$$E(|\{T_k < n \leqslant T_{k+1}: X_n = 0\}|) \sim q^{-1} P(X_{T_k} \approx 0) \tag{10}$$

Finally, if $T_k \approx kq^{-1}$ with probability tending to 1 sufficiently fast as $k \to \infty$, then (9) and (10) together imply (8), because $P(X_n \approx 0) \sim (d/2\pi n)^{d/2}$ as $n \to \infty$ (ref. 9, Section 7).

Of course, the above is only a heuristic explanation. The main difficulty is that a change of the color at the origin mixes up the sequence $(T_k)_{k \geqslant 1}$, hence changes the position occupied by the walk at time T_k, and therefore also affects the local color scenery that is seen by the walk at the start of its kth run, which in turn determines n_k. This means that $E(n_k \mid W)$ and $E(n_k \mid B)$ are not so easy to compare.

In Section 4 we describe numerical calculations of Δ_k, $1 \leqslant k \leqslant 100$, for a range of q values in $d = 1$, 2, and 3 (for Bernoulli coloring and simple random walk). These were performed on finite lattices of up to 10^5 sites with periodic boundary conditions. A color configuration was generated by a random number generator. Then the random walk problem was solved by numerical solution of difference equations (this technique avoids simulation of the walk). The results are plotted in Figs. 1–3, which indeed are seen to confirm (8). The Δ_k for small k exhibit an oscillatory effect: the even Δ_k are smaller than the extrapolation of the odd Δ_k (see Fig. 4). This oscillatory effect damps exponentially fast, after which the power law behavior is found to appear fairly rapidly. For large q the oscillations are strong. It may even happen that $\Delta_{2k} < \Delta_{2k+1}$ for small k when $d \geqslant 2$ (see Fig. 4). For small q, on the other hand, the oscillations are weak, but persist for long times.

Remark that the right-hand side of (5) is an autocorrelation function of a *stationary* process. To make the link with Lorentz models, note that because T_k is a stopping time and because each step of the walk on the average increases the square of its displacement by 1, we have

$$E(X_{T_k}^2) = E(T_k)$$

and hence

$$\delta_k^2 E(X_{T_k}^2) = \delta_k \Delta_k$$

with δ_k the forward difference operator. In view of (5), this identity in a sense is the analogue of the well-known identity $\frac{1}{2}(d/dt)^2 E(X^2(t)) = E(v(0)\,v(t))$ in Lorentz models,[1] valid for a mechanical particle in *equilibrium with its environment*, with $X(t)$ and $v(t)$ its position and velocity at time t. It is believed that $E(v(0)\,v(t)) \sim At^{-d/2-1}$ as $t \to \infty$, with A some model-dependent (negative) amplitude (unknown). Our main result (8) shows that $\delta_k \Delta_k$ has just that behavior.

In a forthcoming publication we prove a rigorous long-time-tail result for a random waiting time model.

2. PROOFS OF THEOREMS 1 AND 2

We shall identify a coloring $(C(z))_{z \in \mathbb{Z}^d}$ with the set $C = \{z \in \mathbb{Z}^d: C(z) = B\}$ containing all its black points. For given C, let

$$n_k(C) = E(n_k \mid C)$$

$$r_k(C) = P(X_n \neq 0 \text{ for all } 0 < n \leqslant T_k,\, X_n = 0 \text{ for some } T_k < n \leqslant T_{k+1} \mid C)$$

i.e., the expected length of the kth run and the probability of first return to the origin during the kth run, both for given C and averaged over the walk ($T_0 = 0$). For $0 \leqslant \eta < 1$ let

$$n(C) = \sum_{k \geqslant 0} \eta^k n_k(C)$$

$$r(C) = \sum_{k \geqslant 0} \eta^k r_k(C)$$

In what follows it is easiest to think of η as a survival probability. If at each visit to a black site the walk has probability $1-\eta$ of being killed and probability η of surviving, then $n(C)$ and $r(C)$ can be interpreted as the expected walk length, resp. the return probability before killing.

To prove Theorem 1, let $0 \notin C$ and observe that the following relation holds:

$$\frac{\kappa e(C)}{1 + \kappa e(C)}\, n(C) = n(C \cup \{0\}) \tag{11}$$

with $\kappa = \eta/(1-\eta)$ and $e(C) = 1 - r(C)$. This is explained as follows. If we change the color of the origin from white to black, then the expected walk length before killing decreases from $n(C)$ to $n(C \cup \{0\})$ because we are introducing an extra killing probability for each time that the walk visits the origin. The expected number of visits to the origin in $C \cup \{0\}$ equals $1 + \eta r(C) + [\eta r(C)]^2 + \cdots$, and so $1 - \eta$ times this sum is the probability that the walk is killed at the origin in $C \cup \{0\}$. Such killing reduces the expected walk length by $n(C)$ and hence we have

$$n(C) = n(C \cup \{0\}) + (1-\eta)\{1 + \eta r(C) + [\eta r(C)]^2 + \cdots\}\, n(C)$$

which is (11).

If we write $P_W(C) = P(C \mid 0 \notin C)$ and $P_B(C) = P(C \mid 0 \in C)$, and average over C, then the right-hand side of (11) gives

$$\begin{aligned}
\int_{C \not\ni 0} n(C \cup \{0\})\, dP_W(C) &\geqslant \int_{C \not\ni 0} n(C \cup \{0\})\, dP_B(C \cup \{0\}) \\
&= \int_{C \ni 0} n(C)\, dP_B(C) \\
&= \sum_{k \geqslant 0} \eta^k \int_{C \ni 0} n_k(C)\, dP_B(C) \\
&= \sum_{k \geqslant 0} \eta^k E(n_k \mid B) \\
&= \kappa q^{-1} \qquad (12)
\end{aligned}$$

where (3) is used in the last equality. The inequality in (12) holds because $n(C)$ is decreasing in C and because $P_B(C \cup \{0\})$ dominates $P_W(C)$ in the sense of Holley.[10] The latter is a consequence of the convexity property of $P(C)$. Thus we get from (11) and (12)

$$\int_{C \not\ni 0} \frac{e(C)\, n(C)}{1 + \kappa e(C)}\, dP_W(C) \geqslant q^{-1} \qquad (13)$$

Note that both (12) and (13) reduce to equalities when the coloring is Bernoulli. We shall need this later to prove (7).

Our next step is to apply the FKG inequality to the left-hand side of (13). Since $e(C)$ is increasing in C, the ratio $e(C)/[1 + \kappa e(C)]$ is increasing in C, and hence the left-hand side is bounded above by the product of

$$\int_{C \not\ni 0} \left\{ \frac{e(C)}{1 + \kappa e(C)} \right\} dP_W(C)$$

and

$$\int_{C \ni 0} n(C)\, dP_W(C)$$

Note here that $P_W(C)$ inherits the convexity property from $P(C)$. The integrand of the first integral is a concave function of $e(C)$ and hence, by Jensen's inequality,

$$\frac{E(e \mid W)\, E(n \mid W)}{1 + \kappa E(e \mid W)} \geqslant q^{-1} \tag{14}$$

where

$$\begin{aligned} E(n \mid W) &= \int_{C \ni 0} n(C)\, dP_W(C) \\ E(r \mid W) &= \int_{C \ni 0} r(C)\, dP_W(C) \\ E(e \mid W) &= 1 - E(r \mid W) \end{aligned} \tag{15}$$

Next we derive a lower bound for $E(r \mid W)$. Fix a walk of n steps and let $l(z, n)$ be its *local time* at site z, i.e.,

$$l(z, n) = |\{0 \leqslant i \leqslant n: X_i = z\}|$$

The probability that the given walk survives its n steps (i.e., is not killed by the black sites it encounters outside the origin) equals

$$\int_{C \ni 0} dP_W(C) \prod_{z \neq 0} (1_{\{z \notin C\}} + \eta^{l(z,n)} 1_{\{z \in C\}})$$

The integrand is a product of decreasing functions in C, and so by FKG this is bounded below by

$$\prod_{z \neq 0} \int_{C \ni 0} dP_W(C)(1_{\{z \notin C\}} + \eta^{l(z,n)} 1_{\{z \in C\}})$$

Another application of FKG shows that this further decreases if $P_W(C)$ is replaced by $P(C)$, and hence the survival probability is bounded below by

$$\prod_{z \neq 0} (1 - q + q\eta^{l(z,n)})$$

which in turn exceeds

$$(1 - q + q\eta)^{\sum_{z \neq 0} l(z,n)}$$

Now use the identity

$$\sum_z l(z, n) = n + 1 \tag{16}$$

It follows that

$$E(r \mid W) \geqslant \sum_{n \geqslant 1} (1 - q + q\eta)^{n-1} P(X_m \neq 0 \text{ for all } 0 < m < n, X_n = 0)$$

because $l(0, n) = 2$ if first return to the origin occurs at time n. Hence

$$E(r \mid W) \geqslant \frac{1}{z} F(z) \Big|_{z = 1 - q + q\eta}$$

with $F(z)$ the generating function for first return to the origin by the random walk. Next use the identity $G(z) = 1/[1 - F(z)]$ (see ref. 9, Section 1). Then (14) yields

$$E(n \mid W) - \kappa q^{-1} \geqslant q^{-1} \frac{zG(z)}{1 - (1 - z)\, G(z)} \Big|_{z = 1 - q + q\eta}$$

Finally, use the identities

$$\begin{aligned} E(n) &= qE(n \mid B) + (1 - q)\, E(n \mid W) \\ E(n \mid B) &= \kappa q^{-1} \end{aligned} \tag{17}$$

with $E(n \mid B)$ and $E(n)$ defined in analogy with (15) to obtain

$$\begin{aligned} E(n_0) + \sum_{k \geqslant 1} \eta^k \Delta_k &= E(n) - E(n \mid B) \\ &= (1 - q)[E(n \mid W) - E(n \mid B)] \\ &\geqslant \frac{1 - q}{q} \frac{zG(z)}{1 - (1 - z) G(z)} \Big|_{z = 1 - q + q\eta} \end{aligned}$$

This is slightly stronger than Theorem 1, because $G(z) \geqslant 1$ for all $z \geqslant 0$. ∎

To prove Theorem 2, first note that the right-hand side of (7) becomes a lower bound after we take the limit $\eta \uparrow 1$ in (6). This lower bound is infinite when the random walk is recurrent, so we need only worry about the transient case. Now let us return to (13), which we know reduces to an equality when the coloring is Bernoulli. Since for every $C \not\equiv 0$ and $0 \leqslant \eta < 1$ we have $r(C) \leqslant F(1) = P(X_n = 0 \text{ for some } n > 0) < 1$, it follows that

$$E(n \mid W) \leqslant \frac{1}{q[1 - F(1)]} + \kappa q^{-1}$$

With (17) this extends to

$$E(n) \leqslant \frac{1-q}{q[1-F(1)]} + \kappa q^{-1}$$

which provides the upper bound because $G(1) = 1/[1-F(1)]$. ∎

3. PROOF OF THEOREM 3

This section has two parts. In Section 3.1 we outline the main arguments, which are based on Lemmas 1–7 below. The proof of Lemmas 1 and 4–6 is given right away. In Section 3.2 we prove Lemmas 2, 3, and 7, which are more technical.

3.1. Main Arguments

Let $H(\eta)$ denote $E(n \mid W) - E(n \mid B)$. Then, as below (17),

$$(1-q)\, H(\eta) = E(n_0) + \sum_{k \geqslant 1} \eta^k \Delta_k \tag{18}$$

It is known that, for Bernoulli coloring and simple random walk, $E(n_0) < \infty$ and $|\Delta_k| \leqslant \Delta_1 < \infty$ for all $k \geqslant 1$ (see footnote 4 and refs. 6 and 7), hence $H(\eta)$ is analytic on $\{\eta \in C \colon |\eta| < 1\}$. We start from the following formal expression, where the expectation is over the walk only:

Lemma 1. For $\eta \in C$ with $|\eta| < 1$

$$H(\eta) = \sum_{n \geqslant 0} E\left((1 - \eta^{l(0,n)}) \prod_{z \neq 0} (1 - q + q\eta^{l(z,n)}) \right) \tag{19}$$

The sum is uniformly convergent on compact subsets.

Proof of Lemma 1. For $0 \leqslant \eta < 1$, Eq. (19) can be understood as follows. If we condition on the origin to be white, then the probability that the walk survives all its black visits during n steps equals [see below (15)]

$$\prod_{z \neq 0} (1 - q + q\eta^{l(z,n)})$$

Here we use the Bernoulli property of the coloring. On the other hand, if we condition on the origin to be black, then its survival probability equals

$$\eta^{l(0,n)} \prod_{z \neq 0} (1 - q + q\eta^{l(z,n)})$$

because now 0 contributes an extra factor $\eta^{l(0,n)}$. Subtracting these two products, averaging over the walk, and summing on n to get $E(n \mid W) - E(n \mid B)$, we have (19).

Alternatively, for $\eta \in C$ with $|\eta| < 1$, one can write

$$\begin{aligned}\sum_{k \geqslant 0} \eta^k n_k &= \sum_{k \geqslant 0} \eta^k \sum_{n \geqslant 0} 1_{\{T_k \leqslant n < T_{k+1}\}} \\ &= \sum_{n \geqslant 0} \eta^{\sum_{m=0}^{n} 1_{\{C(X_m) = B\}}} \\ &= \sum_{n \geqslant 0} \eta^{\sum_z l(z,n) 1_{\{C(z) = B\}}}\end{aligned}$$

and average the summand over the coloring conditioned on a white or a black origin, respectively, thereby obtaining the same two products. Take the average over the walk and pull the sums over k and n, respectively, in front (see below).

To see why the second claim of the lemma is true, argue as follows. For all integers $l \geqslant 0$

$$|1 - q + q\eta^l| \leqslant 1 - q + q\,|\eta|^l \leqslant (1 - q + q\,|\eta|)^{1_{\{l > 0\}}}$$

Let H_n denote the nth term in the right-hand side of (19). Then we have

$$|H_n(\eta)| \leqslant 2E((1 - q + q\,|\eta|)^{R_n - 1})$$

with

$$R_n = \sum_z 1_{\{l(z,n) > 0\}}$$

From the identity (16) it follows that

$$R_n \geqslant (n+1)/\sup_z l(z, n)$$

Hence

$$\begin{aligned}&E((1 - q + q\,|\eta|)^{R_n - 1}) \\ &\qquad \leqslant P(\sup_z l(z, n) \geqslant n^{2/3} + 1) + \exp[-q(1 - |\eta|)n^{1/3}] \qquad \text{for all } n\end{aligned}$$

In Lemma 7 below we shall see that

$$P(\sup_z l(z, n) \geqslant n^{2/3} + 1) \leqslant (n+1)\exp(-C_1 n^{1/3}) \qquad \text{for all } n$$

for some $C_1 > 0$. Hence the sum in (19) converges uniformly on compact subsets of $\{\eta \in C: |\eta| < 1\}$.

It remains to see why the average over the walk may be interchanged with the sums over k and n, respectively, as was claimed above. For $0 \leqslant \eta < 1$ this is allowed because all terms are nonnegative and the r.h.s. of (18) and (19) converge (ref. 11, Theorem 27.F). For $\eta \in C$, $|\eta| < 1$, use dominated convergence (ref. 11, Theorem 26.D.) ∎

Our proof proceeds with two further lemmas based on Lemma 1. We want to show that Δ_k decays like $k^{-d/2}$ for large k. Therefore we are looking for a singularity as $\eta \uparrow 1$ in $H^{(D)}(\eta) = (d/d\eta)^D H(\eta)$ with $D = \lfloor \frac{1}{2} d \rfloor$, of the form $(1-\eta)^{-1/2}$ for d odd and $(1-\eta)^{-1}$ for d even ($\lfloor \cdot \rfloor$ denotes the integer part). This singularity will be found in $G^{(D)}(1-q-q\eta) = (d/d\eta)^D\, G(1-q+q\eta)$.

Lemma 2. $H^{(m)}(\eta)$ converges as $\eta \downarrow -1$, for every $m \geqslant 0$.

Lemma 3. $H^{(D)}(\eta) \sim q^{-1} G^{(D)}(1-q+q\eta)$ as $\eta \uparrow 1$, for every $d \geqslant 1$.

Lemma 3 relates the singularity of $H(\eta)$ at $\eta = 1$ to elementary asymptotics of simple random walk. Lemma 2 shows that there is no singularity at $\eta = -1$. The proof of both lemmas follows later in Section 3.2.

The result needed about simple random walk is as follows.

Lemma 4. For every $d \geqslant 1$

$$G^{(D)}(1-q+q\eta) \sim \Gamma\left(D - \frac{d}{2} + 1\right) q^{-1} \left(\frac{dq}{2\pi}\right)^{d/2} (1-\eta)^{-(D-d/2+1)} \qquad \text{as } \eta \uparrow 1$$

Proof of Lemma 4. The proof can be obtained by a straightforward calculation based on well-known properties of the return to the origin of simple random walk. A more elegant argument goes as follows.

Let $G(1-q+q\eta) = \sum_{k \geqslant 0} g_k \eta^k$. Then it is easy to see that the coefficient g_k can be written as

$$g_k = q^{-1} P(X_{\nu_1 + \cdots + \nu_{k+1} - 1} = 0) \tag{20}$$

where $(\nu_i)_{i \geqslant 1}$ is an i.i.d. sequence of random variables, independent of $(X_n)_{n \geqslant 0}$, with distribution $P(\nu_i = m) = q(1-q)^{m-1}$ ($m = 1, 2,...$). Since $P(X_n = 0) \sim [1 + (-1)^n](d/2\pi n)^{d/2}$ as $n \to \infty$ (ref. 9, Section 7), and since by the Cramér–Chernoff theorem (ref. 12, Theorem 9.3) $P(|\nu_1 + \cdots + \nu_k - kq^{-1}| > \varepsilon k)$ decays exponentially in k for every $\varepsilon > 0$, it follows that

$$g_k \sim q^{-1} \left(\frac{dq}{2\pi k}\right)^{d/2} \qquad \text{as} \quad k \uparrow \infty$$

The result now follows by standard Abelian arguments (ref. 13, Corollary 1.7.3). ▮

From Lemmas 3 and 4 we know the singularity of $H^{(D)}(\eta)$ at $\eta = 1$, while Lemma 2 excludes any contribution from $\eta = -1$. This determines the asymptotic behavior of the Δ_k as $k \to \infty$ because the latter satisfy a *regularity condition.* Indeed, the usual requirement for the application of a Tauberian theorem is that the Δ_k are asymptotically monotone. We are not able to prove this in a direct manner. However, we use the observation that the Δ_k are moments of a *spectral measure* on a compact interval. This property turns out to be a very crucial part of the proof. Without it we would have trouble excluding oscillations and we would have to resort to an analysis of $H(\eta)$ in the neighborhood of $\eta = 1$ and $\eta = -1$ in the complex plane.

Lemma 5. There exists a left-continuous nondecreasing function μ on the interval $[-1, 1]$ such that for all $k \geqslant 1$

$$\Delta_k = \int_{-1}^{1} t^{k-1} \, d\mu(t)$$

Proof of Lemma 5. Recall the notation introduced in Section 2. Let $C \subset \mathbb{Z}^d$ denote a generic color configuration (identified with the set of its black points). Let $\Omega_B = \{C \subset \mathbb{Z}^d \colon 0 \in C\}$ and let $P_B(C) = P(C \mid 0 \in C)$ be Bernoulli coloring on Ω_B. Introduce the *environment process* (EP) $(C_k)_{k \geqslant 1}$ on Ω_B given by

$$C_k = \tau_{X_{T_k}} C \qquad (k \geqslant 1)$$

where C is the color configuration as seen from the origin and τ_x is the shift over x, i.e., $(\tau_x C)(z) = C(\mathbb{Z} + z)$, $z \in x^d$ (recall that $T_1 = 0$ on Ω_B).

Extend the EP to a doubly infinite process $(C_k)_{k \in \mathbb{Z}}$ by running a second independent simple random walk $(X'_n)_{n \geqslant 0}$ from 0, by defining negative black hitting times $(T_k)_{k \leqslant 0}$ as

$$\begin{array}{ll} X'_n \in C & \text{for} \quad n = 0, -T_0, -T_{-1}, \dots \\ X'_n \notin C & \text{otherwise} \end{array}$$

and by putting

$$C_k = \tau_{X_{T_k}} C \qquad (k \leqslant 0)$$

The extended EP is Markov, and is stationary, ergodic,[3] and reversible w.r.t. P_B. The reversibility follows via the symmetry of simple random walk. Also extend $(n_k)_{k\geqslant 1}$ to a doubly infinite process $(n_k)_{k\in\mathbb{Z}}$ by setting

$$n_k = T_{k+1} - T_k \qquad (k \leqslant 0)$$

(with a slight abuse of notation for n_0, which now is $n_0 = T_1 - T_0$ whereas according to our original definition, $n_0 = T_1 = 0$).

Next, consider the Hilbert space $L^2(\Omega_B, \mathbb{R}, P_B)$ with inner product

$$(f, g) = \int_{\Omega_B} f(C)\, g(C)\, dP_B(C)$$

We may then rewite (3) and (4) as

$$q^{-1} = E(n_k \mid B) = E(n_1 \mid B) = (1, s) \qquad (k \in \mathbb{Z})$$
$$q^{-1}E(n_k) = E(n_1 n_{k+1} \mid B) = E(n_0 n_k \mid B) = (s, Q^{k-1}s) \qquad (k \geqslant 1)$$

where $s(C)$ is the average length of a run started from $0 \in C$ and Q is the Markov kernel of the EP acting as operator on Ω_B,

$$(Qf)(C) = \sum_{C' \in \Omega_B} Q(C, C')\, f(C') \qquad (C \in \Omega_B)$$

We have $s \in L^2(\Omega_B, \mathbb{R}, P_B)$ because $(s, s) = q^{-1}E(n_1) < \infty$ (see footnote 4 and refs. 6 and 7). Since Q is self-adjoint by reversibility of the EP, the claim follows from the spectral representation theorem (ref. 14, Chapter VII) applied to (5)

$$\begin{aligned} q^{-1}\Delta_k &= E(n_1 n_{k+1} \mid B) - E(n_1 \mid B)\, E(n_{k+1} \mid B) \\ &= E((n_0 - q^{-1})(n_k - q^{-1}) \mid B) \\ &= ((s - q^{-1}), Q^{k-1}(s - q^{-1})) \end{aligned}$$

The function μ in the lemma may have jumps anywhere on $[-1, 1)$. The integral in the lemma does not cover a jump at 1 because μ is taken to be left-continuous. However, ergodicity of the EP precludes a jump at 1 [because $(s - q^{-1}, 1) = 0$]. ∎

From Lemma 5 and (18) we obtain

$$\begin{aligned} (1-q)\, H(\eta) &= E(n_0) + \sum_{k \geqslant 1} \eta^k \int_{-1}^{+1} t^{k-1}\, d\mu(t) \\ &= E(n_0) + \eta \int_{-1}^{+1} \frac{1}{1-\eta t}\, d\mu(t) \qquad (|\eta| < 1) \end{aligned}$$

By a simple transformation (see below) the integral can be written as a Stieltjes transform, for which the following Tauberian theorem holds:

Tauberian Theorem. Let $m>0$ and let α be a right-continuous nondecreasing function on the interval $[0, 2]$, $\alpha(0)=0$. Assume that

$$f(s)=\int_0^2 (s+t)^{-m}\, d\alpha(t)$$

converges for $s>0$. Then for any $A \geqslant 0$ and $0<\gamma<m$ the following are equivalent:

$$\alpha(t) \sim A\,\frac{\Gamma(m)}{\Gamma(\gamma+1)\,\Gamma(m-\gamma)}\,t^{\gamma} \qquad \text{as} \quad t \downarrow 0$$

$$f(s) \sim A s^{\gamma-m} \qquad \text{as} \quad s \downarrow 0$$

Proof. The Abelian direction of the proof follows immediately from ref. 15, Chapter V, Theorem 2.a and Corollary 2.a. The Tauberian direction follows from ref. 13, Theorem 1.7.4. Indeed, let $\beta(u)=0$ if $u \leqslant 1/2$ and $\beta(u)=\int_{1/u}^{2} t^{-m}\, d\alpha(t)$ if $u>1/2$. Then from $\int_{1/2}^{\infty} (s^{-1}+u)^{-m}\, d\beta(u)= s^m f(s) \sim A s^{\gamma}$ as $s \downarrow 0$, it follows that $\beta(u) \sim A[\Gamma(m)/\Gamma(\gamma)\,\Gamma(m-\gamma+1)]\,u^{m-\gamma}$ as $u \uparrow \infty$ by the theorem cited. In combination with $\alpha(t)=\int_{1/t}^{\infty} u^{-m}\, d\beta(u)$, the latter implies the desired asymptotic behavior of $\alpha(t)$ (by standard Abelian arguments again). ▌

The choice of the interval $[0, 2]$ in the above Tauberian theorem is arbitrary and is made for convenience. Application to the expression for $(1-q)\,H(\eta)$ leads, in combination with Lemmas 2–4, to the following result:

Lemma 6. For every $d \geqslant 1$

$$\mu(1)-\mu(t) \sim \frac{(1-q)q^{d/2-2}(d/2\pi)^{d/2}}{\Gamma(d/2+1)}\,(1-t)^{d/2} \qquad \text{as} \quad t \uparrow 1$$

For every $m \geqslant 0$

$$\lim_{t \downarrow -1} (1+t)^{-m}[\mu(t)-\mu(-1)]=0$$

Proof of Lemma 6. First consider $D=0$ (or $d=1$). Make the transformation $\alpha(1-t)=\mu(1)-\mu(t)$ to rewrite

$$(1-q)\,H(\eta)=E(n_0)+\int_0^2 \left(\frac{1-\eta}{\eta}+t\right)^{-1} d\alpha(t)$$

From Lemmas 3 and 4

$$(1-q)\,H(\eta) \sim A\left(\frac{1-\eta}{\eta}\right)^{-1/2} \qquad \text{as} \quad \eta \uparrow 1$$

with $A = \Gamma(1/2)(1-q)q^{-3/2}(2\pi)^{-1/2}$. Combine the last two displays and apply the Tauberian theorem with $m=1$, $\gamma=1/2$, and $s=(1-\eta)/\eta$. This gives

$$\alpha(t) \sim A\,\frac{\Gamma(1)}{\Gamma(3/2)\,\Gamma(1/2)}\,t^{1/2} \qquad \text{as} \quad t \downarrow 0$$

which is the first claim of the lemma for $d=1$.

The same argument works for $D>0$ (or $d \geqslant 2$). By an easy computation

$$(1-q)\,H^{(D)}(\eta) = \Gamma(D+1)\eta^{-D-1}\int_0^2 \left(\frac{1-\eta}{\eta}+t\right)^{-D-1} (1-t)^{D-1}\,d\alpha(t)$$

This is a Stieltjes transform except for the innocent factors η^{-D-1} and $(1-t)^{D-1}$, which both have limit 1. Again, from Lemmas 3 and 4

$$(1-q)\,H^{(D)}(\eta) \sim A'\left(\frac{1-\eta}{\eta}\right)^{-(D-d/2+1)} \qquad \text{as} \quad \eta \uparrow 1$$

with

$$A' = \Gamma(D-d/2+1)(1-q)q^{d/2-2}(d/2\pi)^{d/2}$$

Now apply the Tauberian theorem with $A = A'/\Gamma(D+1)$, $m=D+1$, $\gamma = d/2$, and $s=(1-\eta)/\eta$. This proves the first claim for $d \geqslant 2$.

The second claim follows similarly. Make the transformation $\alpha(1+t) = \mu(t) - \mu(-1)$, take the right-continuous version of α, and use Lemma 2. Use the Tauberian theorem with $A=0$. ∎

Lemma 6 identifies the singularity in the spectral measure μ. Finally, we can apply the standard Abelian theorem for Laplace transforms (ref. 15, Chapter V, Theorem 1 and Corollary 1a) to deduce the asymptotic behavior of Δ_k from Lemma 5. This completes the proof of Theorem 3. ∎

Note that our long-time-tail result can be written as

$$(1-q)^{-1}\Delta_k \sim q^{-1}g_k \tag{21}$$

Expressions (21) and (20) are the rigorous versions of (9) and (10).

3.2. Proof of Lemmas 2 and 3

In the proof of Lemmas 2 and 3 the following large-deviation estimate for $l(z, n)$ will be instrumental (Lemma 7 has been used already in the proof of Lemma 1).

Lemma 7. For arbitrary random walk in $d \geqslant 1$, there exists $K > 0$ such that for all positive integer $h(n)$ and all n

$$P(\sup_z l(z, n) > h(n)) \leqslant (n+1) \exp[-Kh^2(n)/n] \tag{22}$$

Proof of Lemma 7. The left-hand side of (22) is bounded above by

$$\sum_{0 \leqslant i \leqslant n} P(l(X_i, n) > h(n),\, X_j \neq X_i \text{ for } 0 \leqslant j < i) \leqslant (n+1)\, P(l(0, n) > h(n))$$

Let σ_m $(m \geqslant 1)$ denote the time at which the walk returns to the origin for the mth time. By the Markov inequality

$$P(l(0, n) > h(n)) = P(\sigma_{h(n)} \leqslant n) \leqslant \inf_{\xi > 0} \exp\{\xi n\}\, E(\exp\{-\xi\sigma_1\})^{h(n)}$$

because σ_m is a sum of m independent copies of σ_1. Via the identity $G(z) = 1/[1 - F(z)]$ already used earlier, we have

$$E(\exp\{-\xi\sigma_1\}) = F(\exp\{-\xi\}) = 1 - G^{-1}(\exp\{-\xi\})$$

Now use the fact that for arbitrary random walk in $d \geqslant 1$ there exists $A_1 > 0$ such that $P(X_n = 0) \leqslant A_1 n^{-1/2}$ for all n (ref. 9, Section 7). This gives

$$E(\exp\{-\xi\sigma_1\}) \leqslant \exp(-A_2\xi^{1/2}) \qquad \text{for all} \quad \xi > 0$$

for some $A_2 > 0$. Equation (22) follows by taking $\xi = [A_2 h(n)/2n]^2$, the value where the bound obtains its infimum. ∎

Proof of Lemma 2. First consider $m = 0$. Let $\eta = -r$, $0 \leqslant r < 1$. For $\delta > 0$ let V^δ be the set of integers

$$V^\delta = \{l \geqslant 0\colon |1 - q + q(-r)^l| \leqslant 1 - \delta\}$$

There exists $B_1 > 0$ independent of r such that if $\delta \leqslant B_1$ then

$$l \in V^\delta \qquad \text{for } l \text{ odd} \tag{23}$$

Let

$$\Gamma = \{z \in \mathbb{Z}^d\colon z_1 + \cdots + z_d = \text{even}\}$$

and let $H_n(\eta)$ denote the nth term in the right-hand side of (19). Then

$$|H_n(-r)| \leqslant 2E((1-\delta)^{Y_n}) \tag{24}$$

with

$$Y_n = |\{z \in \Gamma \backslash \{0\}: l(z, n) \in V^\delta\}|$$

The aim is to show that Y_n is large with large probability.

Consider the imbedded random walk $(\tilde{X}_i)_{i \geqslant 0}$ on Γ that is defined as follows:

$$\begin{aligned} &\tau(0) = 0 \\ &\tau(i+1) = \inf\{n > \tau(i): X_n \in \Gamma, X_n \neq X_{\tau(i)}\} \qquad (i \geqslant 0) \\ &\tilde{X}_i = X_{\tau(i)} \qquad (i \geqslant 0) \end{aligned}$$

That is, the original random walk is observed only when it visits the set Γ at a site different from the site of its previous visit. The increments $\tau(i+1) - \tau(i)$ are i.i.d. geometric on the positive even integers, have mean $4d/(2d-1)$, and are independent of $(\tilde{X}_i)$. Let $\tilde{l}(z, i)$ denote the local time at $z \in \Gamma$ of the imbedded random walk after i steps. Then

$$l(z, \tau(i+1) - 1) = \tilde{l}(z, i) + \sum_{j=1}^{\tilde{l}(z,i)} \rho(z, j) \qquad (i \geqslant 0, z \in \Gamma) \tag{25}$$

Here $\rho(z, j)$ counts the number of immediate returns by the original random walk to the site $z \in \Gamma$ (in two steps) following the jth visit to z by the imbedded random walk. The $\rho(z, j)$ are i.i.d. geometric on the non-negative integers, have mean $1/(2d-1)$, and are independent of $(\tilde{X}_i)$ and hence also of $(\tilde{l}(z, i))$. For any $B_2 > 0$, write

$$\begin{aligned} P(Y_n \leqslant B_2 n^{1/3}) &= \sum_{i \geqslant 0} P(\tau(i) \leqslant n < \tau(i+1), Y_n \leqslant B_2 n^{1/3}) \\ &\leqslant \sum_{i \geqslant 0} P(\tau(i) \leqslant n < \tau(i+1), Y_{\tau(i+1)-1} \leqslant B_2 n^{1/3} + 1) \\ &\leqslant \sum_{i \leqslant n/8} P(n < \tau(i+1)) + \sum_{i > n/8} P(Y_{\tau(i+1)-1} \leqslant 2B_2 i^{1/3} + 1) \end{aligned} \tag{26}$$

The first term in the right-hand side decays exponentially in n by the Cramér–Chernoff theorem (ref. 12, Theorem 9.3) because $E(\tau(i+1) - \tau(i)) = 4d/(2d-1) \leqslant 4$. To estimate the second term, proceed as follows.

Because the geometric distribution is aperiodic, it is true that for each $z \in \Gamma$

$$\inf_{l > 0} P\left(l + \sum_{j=1}^{l} \rho(z, j) \in V^{\delta}\right) = \zeta \qquad \text{for some} \quad \zeta > 0$$

Here we use (23), which implies that

$$\liminf_{l \to \infty} P\left(l + \sum_{j=1}^{l} \rho(z, j) \in V^{\delta}\right) \geqslant \frac{1}{2}$$

It now follows from (25) and the definition of Y_n that

$$P(Y_{\tau(i+1)-1} \leqslant 2B_2 i^{1/3} + 1) \leqslant P(Z_1 + \cdots + Z_{\tilde{R}_i - 1} \leqslant 2B_2 i^{1/3} + 1) \quad (27)$$

where

$$\tilde{R}_i = \sum_{z \in \Gamma} 1_{\{\tilde{l}(z,i) > 0\}}$$

$$(Z_m)_{m \geqslant 1} \quad \text{i.i.d.} \quad \text{with} \quad P(Z_1 = 1) = 1 - P(Z_1 = 0) = \zeta$$

Here we rely on the independence of the $\rho(z, j)$ for different z. From $\sum_{z \in \Gamma} \tilde{l}(z, n) = n + 1$ [recall (16)] it follows that

$$\tilde{R}_i \geqslant (i+1)/[\sup_{z \in \Gamma} \tilde{l}(z, i)]$$

Hence from Lemma 7 with $h(n) = \lfloor n^{2/3} \rfloor$

$$\begin{aligned} &P(Z_1 + \cdots + Z_{\tilde{R}_i - 1} \leqslant 2B_2 i^{1/3} + 1) \\ &\qquad \leqslant \exp\{-B_3 i^{1/3}\} + P(Z_1 + \cdots + Z_{i^{1/3}} \leqslant 2B_2 i^{1/3} + 1) \end{aligned} \quad (28)$$

for some $B_3 > 0$. If $2B_2 < \zeta$, then the last term tends to zero as $\exp\{-B_4 i^{1/3}\}$ for some $B_4 > 0$. Combining (26)–(28) and returning to (24) with $\delta \leqslant B_1$, we conclude that

$$|H_n(-r)| \leqslant 2 \exp(-B_5 n^{1/3})$$

for some $B_5 > 0$. Since this bound is independent of r and is summable, and since for all n finite $H_n(-r)$ converges as $r \uparrow 1$ [note that $H_n(\eta)$ is a polynomial in η of degree $n+1$], Lemma 2 follows for $m = 0$.

It is now easy to prove Lemma 2 for $m > 0$. The key is the following inequality:

$$\begin{aligned} &\left| (d/d\eta)^m \prod_{z \neq 0} (1 - q + q\eta^{l(z,n)}) \right| \\ &\qquad \leqslant \sum_k C_m(k) \prod_{z \in S(k)} [l(z, n)^{k_z} 1_{\{k_z \leqslant l(z,n)\}}] \left| \prod_{z \notin S(k) \cup \{0\}} (1 - q + q\eta^{l(z,n)}) \right| \end{aligned} \quad (29)$$

where the sum runs over all functions k: $\mathbb{Z}^d \backslash \{0\} \to \mathbb{N} \cup \{0\}$, and $S(k) = \{z: k_z > 0\}$, and $C_m(k) = m!/\prod_z (k_z!)$ if $\sum_z k_z = m$ and $C_m(k) = 0$ otherwise. For the last factor of (29) the same estimate $2\exp(-B_5 n^{1/3})$ holds. To see this, repeat the proof with Γ replaced by $\Gamma \backslash S(k)$ in the definitions of Y_n and $\tilde{R}_i$. All estimates carry over. Use that

$$\sum_{z \in \Gamma \backslash S(k)} \tilde{l}(z, i) \geqslant i + 1 - |S(k)|\, i^{2/3}$$

when $\sup_{z \in \Gamma} \tilde{l}(z, i) \leqslant i^{2/3}$. Use (16) to estimate

$$\prod_{z \in S(k)} l(z, n)^{k_z} \leqslant n^m \qquad \text{for all } k$$

and

$$\sum_k C_m(k) \prod_{z \in S(k)} 1_{\{k_z \leqslant l(z,n)\}} \leqslant m!\, n^m$$

Hence there exists B_6 independent of r such that

$$|H_n^{(m)}(-r)| \leqslant B_6 n^{2m} \exp(-B_5 n^{1/3}) \tag{30}$$

which is summable. We conclude that $H^{(m)}(-r)$ is bounded and hence converges as $r \uparrow 1$.

In (19) the mth derivative may first be interchanged with the sum, because of the uniform convergence on compact subsets of $\{\eta \in C: |\eta| < 1\}$ (ref. 16, Theorem 8.19), and then be interchanged with the average because $H_n(\eta)$ is a polynomial in η. ∎

Proof of Lemma 3. Set $\varepsilon = 1 - \eta$. First we prove Lemma 3 for $D = 0$, i.e., $d = 1$. Let

$$f(n) = \lfloor n^\alpha \rfloor, \qquad g(n) = \lfloor n^\beta \rfloor, \qquad h(n) = \lfloor n^\gamma \rfloor$$

with α, β, and γ positive constants to be chosen along the way. Suppose that

(i) $$\gamma > \tfrac{1}{2}$$

Then by Lemma 7

$$H(\eta) = \sum_{n \geqslant 0} E\left((1 - \eta^{l(0,n)}) \prod_{z \neq 0} (1 - q + q\eta^{l(z,n)})\, 1_{\{\sup_z l(z,n) \leqslant h(n)\}} \right) + O(1) \tag{31}$$

Split the sum over n into three parts,

$$H(\eta) = H_1(\eta) + H_2(\eta) + H_3(\eta) + O(1)$$

running over $[0, f(\varepsilon^{-1})]$, $(f(\varepsilon^{-1}), g(\varepsilon^{-1}))$, and $[g(\varepsilon^{-1}), \infty)$, respectively. This requires

(ii) $$\alpha \leqslant \beta$$

In $H_1(\eta)$ substitute the expansions

$$\frac{1-\eta^{l(0,n)}}{1-q+q\eta^{l(0,n)}} = \varepsilon l(0,n) + O(\varepsilon^2 l^2(0,n)) = \varepsilon l(0,n)[1 + O(\varepsilon h(n))] \tag{32}$$

$$\begin{aligned}\prod_z (1-q+q\eta^{l(z,n)}) &= \prod_z \exp\{-q\varepsilon l(z,n) + O(\varepsilon^2 l^2(z,n))\} \\ &= \exp\{-q\varepsilon(n+1)[1 + O(\varepsilon h(n))]\}\end{aligned} \tag{33}$$

This uses the uniform bound on $l(z, n)$ as well as the identity (16). Next suppose that

(iii) $$\alpha\gamma < 1$$

Then $\lim_{\varepsilon \downarrow 0} \varepsilon h(f(\varepsilon^{-1})) = 0$, and it follows that

$$H_1(\eta) = \varepsilon[1 + o(1)] \sum_{0 \leqslant n \leqslant f(\varepsilon^{-1})} E(l(0,n)) \exp\{-q\varepsilon n[1 + o(1)]\} \tag{34}$$

where we again use (i) and Lemma 7 to get rid of the indicator afterward. Note that in (34) the $1 + o(1)$ factor depends on n and ε, but tends to zero as $\varepsilon \downarrow 0$ uniformly in n over the sum.

To estimate $H_2(\eta)$, observe that $\prod_{z \neq 0} (1-q+q\eta^{l(z,n)})$ is decreasing in n because with each step of the walk one of the local times must increase (use that η is positive). Therefore, bounding this product for each n in $H_2(\eta)$ by its estimate at $n = f(\varepsilon^{-1})$ just obtained in $H_1(\eta)$, we get

$$H_2(\eta) = O(g(\varepsilon^{-1}) \exp\{-\tfrac{1}{2} q\varepsilon f(\varepsilon^{-1})\}) \tag{35}$$

To estimate $H_3(\eta)$, use that for all integers $l \geqslant 0$

$$|1-q+q\eta^l| \leqslant 1-q+q\,|\eta|^l \leqslant (1-q+q\,|\eta|)^{1\{l>0\}}$$

to bound

$$\left|\prod_{z \neq 0} (1-q+q\eta^{l(z,n)})\right| \leqslant (1-q+q\,|\eta|)^{R_n - 1}$$

where $R_n = \sum_z 1_{\{l(z,n)>0\}}$, as in the proof of Lemma 1. By the inequality $R_n \geqslant (n+1)/\sup_z l(z,n)$, this yields

$$H_3(\eta) = O\left(\sum_{g(\varepsilon^{-1}) \leqslant n < \infty} \exp\left\{-\frac{1}{2}\frac{q\varepsilon n}{h(n)}\right\}\right) \tag{36}$$

again because of the uniform bound on $l(z, n)$. Finally, suppose that

(iv) $\alpha > 1, \qquad \beta(1-\gamma) > 1$

Then $H_2(\eta) = o(1)$ and $H_3(\eta) = o(1)$ by (35) and (36), and also

$$\varepsilon \sum_{f(\varepsilon^{-1}) < n < \infty} \exp\{-\tfrac{1}{2}q\varepsilon n\}\, E(l(0,n)) = o(1)$$

so that, via (34),

$$H(\eta) = \varepsilon[1+o(1)] \sum_{n \geqslant 0} E(l(0,n)) \exp\{-q\varepsilon n[1+o(1)]\} + O(1) \tag{37}$$

But

$$\begin{aligned}\sum_{n\geqslant 0} \exp\{-q\varepsilon n\}\, E(l(0,n)) &= \sum_{n\geqslant 0} \exp\{-q\varepsilon n\} \sum_{m=0}^{n} P(X_m = 0)\\ &= (1-\exp\{-q\varepsilon\})^{-1}\, G(\exp\{-q\varepsilon\})\end{aligned}$$

Moreover, from Lemma 4 we have

$$G(\exp\{-q\varepsilon[1+o(1)]\})/G(\exp\{-q\varepsilon\}) = 1 + o(1)$$

and so we conclude that

$$H(\eta) \sim q^{-1} G(1-q+q\eta) \qquad (\eta \uparrow 1)$$

In this last step we use that simple random walk in $d=1$ is recurrent, so that the right-hand side diverges and the error terms are of higher order. This proves Lemma 3 for $D=0$ (or $d=1$) because conditions (i)–(iv) can be made to hold, for instance, by choosing $\alpha = 5/4$, $\beta = 4$, and $\gamma = 2/3$.

It is now easy to prove Lemma 3 for $D>0$ (or $d \geqslant 2$). The key is (2.29). As we saw before, by taking the Dth derivative of (19) we add at most a factor $O(n^{2D})$ to the summand. But with α, β, and γ as in (i)–(iv) we saw that the summands in $H_2(\eta)$ and $H_3(\eta)$ are exponentially small in some fractional power of ε^{-1} [see (35) and (36)], and therefore our previous estimates can easily accommodate the extra factor n^{2D}. Thus,

$$H_2^{(D)}(\eta) = o(1)$$

$$H_3^{(D)}(\eta) = o(1)$$

and hence

$$H^{(D)}(\eta) = H_1^{(D)}(\eta) + O(1)$$
$$= (d/d\eta)^D\{[1+o(1)]q^{-1}G(1-q+q\eta)\} + O(1) \qquad (38)$$

Finally, pull $1+o(1)$ in front of the differentiation. This step is not trivial. However, it is easily justified by inspecting how the $o(1)$ term arises in (32) and (33), and by verifying that each derivative of the $o(1)$ term adds at most a singularity $\varepsilon^{-\alpha\gamma}$ $(\alpha\gamma < 1)$ while each derivative of the main factor adds ε^{-1}. Here we ask the reader to check the details. ∎

4. NUMERICAL CALCULATIONS

We now report the results of our numerical study for Bernoulli coloring and simple random walk in $d=1$, 2, and 3. Our model contains two random elements: the coloring and the walk. We simulate a single color configuration on a large finite box $\Lambda \subset Z^d$ by use of a pseudo-random-number generator, coloring each site of Λ black with probability q. If Λ is large enough, then most local color environments are present with the correct statistics. We have avoided simulation of the walk. Instead, given the color configurations, we calculate the relevant average quantities via numerical solution of difference equations with periodic boundary conditions. These equations implicitly average over the walk as well as over its starting position (the latter replaces stationarity of the coloring). This procedure is repeated for several realizations of the coloring. It allows us to assess the dependence on the configuration and, if necessary, to take the average over the configurations generated.

In this way we have obtained results that are accurate within numerical precision. It would have been easier also to simulate the walk. However, we want to draw quantitative conclusions about Δ_k for $1 \leqslant k \leqslant 100$, and since typical values are $\Delta_{100} \sim 10^{-3}$, the numerical computation of $E(n_k)$ should at least be accurate up to about 10^{-5} to get some relative precision. Such accuracy cannot be achieved by means of simulation of the walk.

The method of studying random walk problems by means of numerical solution of difference equations has been applied successfully in the past.[17,18] Here the novel aspect is that the transition matrix between black sites is itself the solution of difference equations, so that the method is applied twice: first to calculate this matrix and then to iterate it.

4.1. Basic Quantities and Equations

Let $C \subset \Lambda$ be the set of black sites in the box Λ. The following three quantities are calculated:

$P_i^{(1)}$ = probability that i is the *first* black site hit when the random walk starts from a random position ($i \in C$)

P_{ij} = probability that j is the *next* black site hit if the random walk starts from black site i ($i, j \in C$)

s_i = average number of steps needed to hit any *next* black site if the random walk starts from black site i ($i \in C$)

With this notation we have

$$E(n_k) = \sum_{i \in C} P_i^{(k)} s_i \qquad (k \geqslant 1) \tag{39}$$

with

$$P_j^{(k)} = \sum_{i \in C} P_i^{(k-1)} P_{ij} \qquad (k > 1,\ j \in C) \tag{40}$$

Since simple random walk is symmetric, we have, in addition, the relation

$$P_i^{(1)} = \frac{1}{|\Lambda|} s_i \qquad (i \in C) \tag{41}$$

as is easily shown by reversing time. As a consequence, only two quantities, P_{ij} and s_i, need to be calculated.

The matrix of transition probabilities $(P_{ij})_{i,j \in C}$ and the vector of average run lengths $(s_i)_{i \in C}$ are calculated by solving difference equations involving two related quantities:

Q_{ij} = probability that j is the *first* black site hit if the random walk starts from site i ($i \in \Lambda$, $j \in C$)

t_i = average number of steps needed to hit any *first* black site if the random walk starts from site i ($i \in \Lambda$)

Let Ω be the discrete Laplace operator on Λ defined by

$$\Omega_{ii} = -1$$

$$\Omega_{ij} = \frac{1}{2d} \qquad \text{if} \quad |i-j| = 1$$

$$\Omega_{ij} = 0 \qquad \text{otherwise}$$

Then (Q_{ij}) and (t_i) are the solutions of the following sets of equations:

$$\sum_{j \in \Lambda} \Omega_{ij} t_j = -1 \qquad (i \in \Lambda \backslash C) \tag{42}$$

$$\sum_{j \in \Lambda} \Omega_{ij} Q_{jk} = 0 \qquad (i \in \Lambda \backslash C,\ k \in C) \tag{43}$$

with boundary conditions $Q_{ij} = \delta_{ij}$ and $t_i = 0$ for all $i, j \in C$. The uniqueness of these solutions follows from an easy lemma.

Lemma. If $C \subset \Lambda$ is nonempty, then the set of equations $\sum_{j \in \Lambda} \Omega_{ij} x_j = 0$ $(i \in \Lambda \backslash C)$ with boundary condition $x_j = 0$ $(j \in C)$ has the unique solution $x_j \equiv 0$ $(j \in \Lambda)$.

Proof. Let $M = \max_{j \in \Lambda} |x_j|$. If $C \neq \Lambda$ then there exists $m \in \Lambda \backslash C$ such that $M = |x_m|$. It follows that

$$M = |x_m| = \left| \frac{1}{2d} \sum_{j:\,|j-m|=1} x_j \right| \leqslant \frac{1}{2d} \sum_{j:\,|j-m|=1} |x_j| \leqslant M$$

Hence $x_j = x_m$ for all neighbors j of m. If $j \in \Lambda \backslash C$, then we can repeat the argument, and so x_j is constant along any path of connections between pairs of sites of which one lies in $\Lambda \backslash C$. But Λ is connected and C is nonempty, and so $M = 0$. ∎

Once we have computed (Q_{ij}) and (t_i), we can calculate

$$P_{ij} = \frac{1}{2d} \sum_{k:\,|k-i|=1} Q_{kj}$$

$$s_i = 1 + \frac{1}{2d} \sum_{k:\,|k-i|=1} t_k$$

and so via (39)–(41) the numerical solution of (42) and (43) leads to the evaluation of $E(n_k)$.

The numerical solution of (42) and (43) can be obtained by using standard algorithms. The choice of algorithm is somewhat restricted because the number of variables is large ($|\Lambda| \sim 10^5$). It turned out that a simple iteration scheme worked in all cases and we did not encounter any problems of convergence.

4.2. Results

In $d = 1$ both (P_{ij}) and (s_i) can be calculated analytically and (P_{ij}) contains mostly zeros. This is because from any black site only transitions to the two closest black sites are possible. This simplifies the numerical calculations considerably, since (39)–(41) can be iterated directly. Our numerical results in $d=1$ are therefore more accurate than in higher dimensions. Configurations of 10^5 sites with $q=0.1$–0.9 could be examined. Each configuration required a few minutes of CPU time on a VAX-8200. A power law behavior $\Delta_k \sim Ak^{-1/2}$ was found for k large, with A slightly dependent on the configuration (fluctuations of less than 1%). We averaged Δ_k, $1 \leqslant k \leqslant 100$, over 100 configurations for $q=0.1$, 0.2, and 0.5 plotted the results in Fig. 1. There is clear evidence for the asymptotic behavior $\Delta_k \sim Ak^{-1/2}$ with $A=(1-q)q^{-3/2}(1/2\pi)^{1/2}$. In fact, the numerical values are slightly lower than the analytic expression. This is caused by finite-size effects, i.e., large color fluctuations do not get the correct weight even over 100 color realizations. We found that the power law is reached faster when q gets closer to 0.5. Also, the even–odd oscillatory effect is less pronounced when q is close to 0.5. For the densities we investigated, the

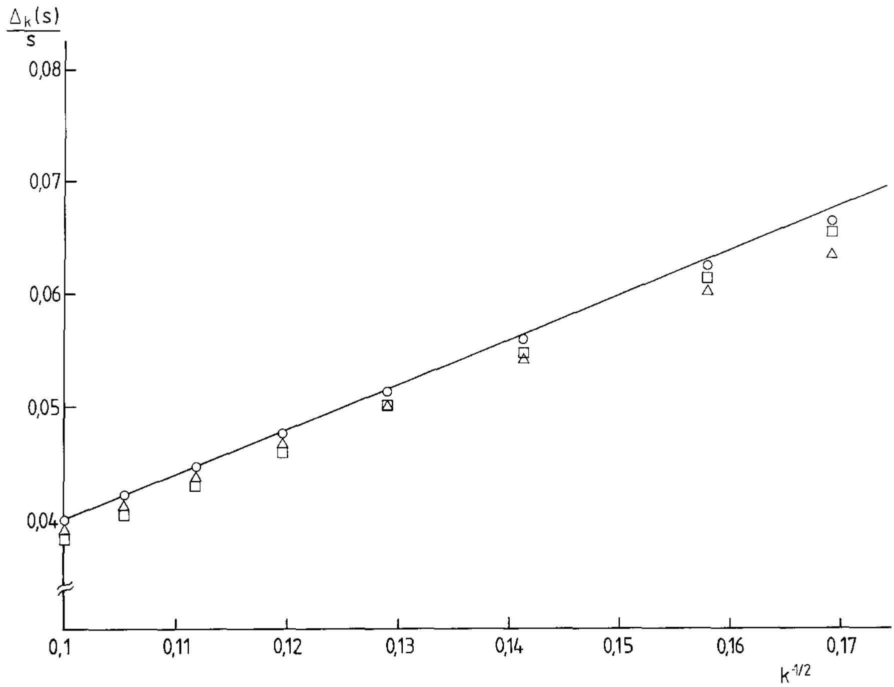

Fig. 1 ($d=1$). Δ_k/S with $S=(1-q)/q^{3/2}$ plotted as a function of $k^{-1/2}$. The line is (8). The triangular, circular, and rectangular dots are the numerical results for $q=0.1$, $q=0.2$, and $q=0.5$, respectively. The figure shows that for large enough k the data are on a straight line, are independent of q, and are either on or slightly below the line shown.

power law was found for $k \geqslant 30$ and the even–odd effect was clearly seen for k up to 10. In all simulations Δ_k was found to be monotone in k.

For $d > 1$ one has to resort to numerical evaluation of (P_{ij}) and (s_i) as explained in Section 4.1. Because (P_{ij}) contains many nonzero elements, a large amount of computing time and computer memory is needed. There-

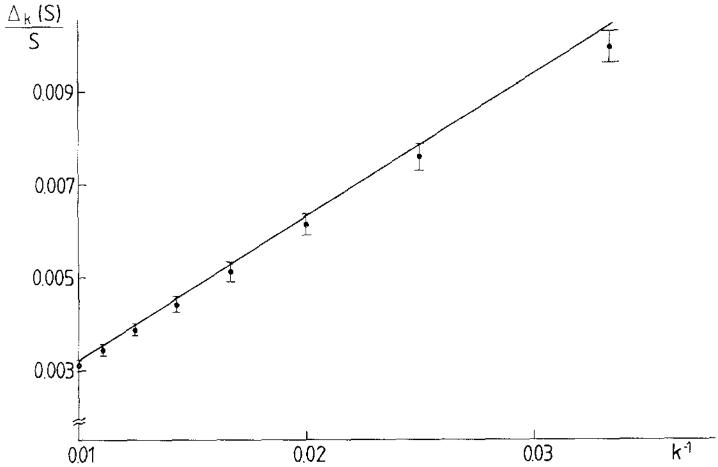

Fig. 2 ($d=2$). Δ_k/S with $S=(1-q)/q$ plotted as a function of k^{-1}. The line is (8). The dots are the numerical results for $q=0.7$. The error bars indicate the statistical error. The same observations as in the one-dimensional case can be made.

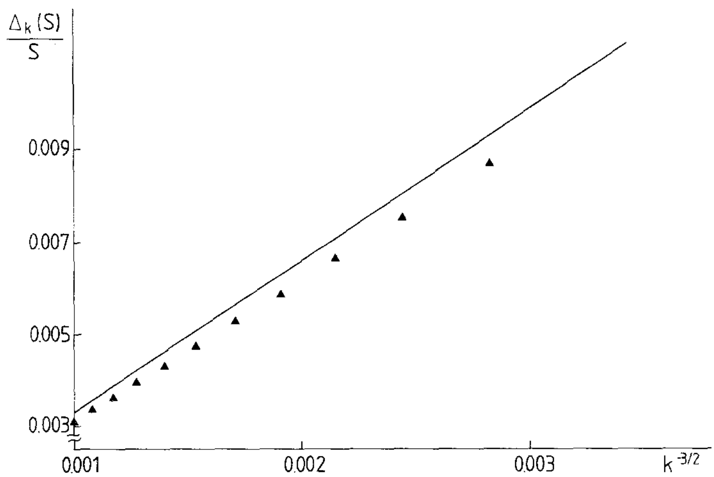

Fig. 3 ($d=3$). Δ_k/S with $S=(1-q)/q^{1/2}$ plotted as a function of $k^{-3/2}$. The line is (8). The triangular dots are the numerical results for $q=0.8$. The error bars are not drawn, but are of a size slightly larger than in Fig. 2.

fore we performed our calculations on a supercomputer (Cyber 205, Amsterdam).

In $d = 2$, configurations of $(200)^2$ sites are needed to see the power law for densities $q = 0.5$–0.9 (with fluctuations in A of a few percent). In Fig. 2 our results are plotted for $q = 0.7$ with Δ_k, $1 \leqslant k \leqslant 100$, averaged over 10 color configurations. In $d = 3$ the power law is seen for densities $q = 0.7$–0.9 and configurations of $(35)^3$ sites (with again fluctuations in A of a few percent). Figure 3 plots Δ_k, $1 \leqslant k \leqslant 100$, for $q = 0.8$ averaged over 10 color configurations. Both in $d = 2$ and $d = 3$ there is clear evidence for (8). Because of the high densities to which our simulations are restricted, the even–odd effect is rather pronounced. We even found that $\Delta_{2k} < \Delta_{2k+1}$ for small k. This effect is visible for k up to 20 (see Fig. 4). The power law was found for $k \geqslant 50$.

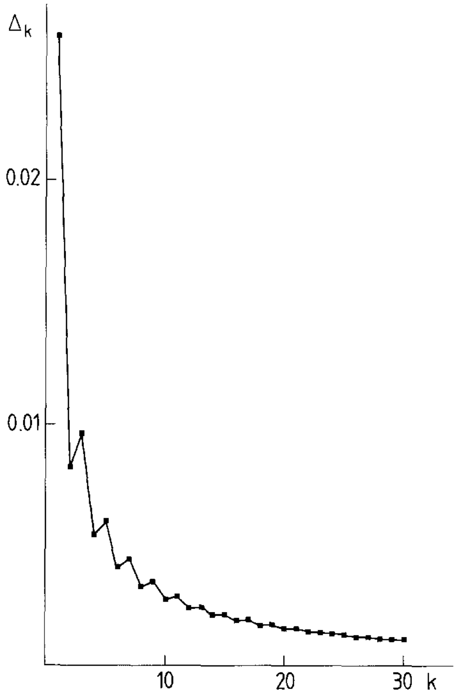

Fig. 4. The first Δ_k as a function of k. The dots are the numerical data. The density is $q = 0.9$, the dimension is $d = 3$. Note the oscillatory effect, which damps exponentially fast.

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